THE CAUCHY-LIPSCHITZ METHOD FOR INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS.*

By William L. Hart.

1. **Introduction.**—The present paper is concerned with the solution of a denumerably infinite system of ordinary differential equations, in which the variables and functions assume real values,

$$(1) \qquad \frac{dx_i}{dt} = f_i(\xi,t) \qquad [i = 1, 2, \cdots; \ \xi = (x_1, \cdots, x_n, \cdots)].$$

Hypotheses will be imposed on (1) under which it will be proved that *a least* one solution exists satisfying given initial conditions.

In a previous paper† the author stated certain theorems concerning completely continuous functions of infinitely many variables, and considered a system (1) in which the f_i were of this type. The unique existence of a solution of (1), satisfying given initial conditions, was established by a generalization of the Picard approximation process. It was found necessary to assume that the f_i satisfied a condition with respect to ξ analogous to the Lipschitz condition used in the consideration of finite systems of differential equations. In the discussion which follows, the system (1) will be considered without the assumption of the Lipschitz condition and, in analogy with the corresponding result for finite systems,‡ it will be established that at least one solution exists satisfying given initial conditions. The proof will be made by use of an extension to infinite systems of the notion of Cauchy polygons, and will be similar in method to that used by Montel§ in establishing the corresponding result for finite systems.

The general system of notation used below is vectorial in character. After any notation has been introduced, it will retain the same meaning as a formal algebraic expression whenever used in the future.

2. **Extension of a Theorem of Arzelà.**—Let R denote the region of points $\xi = (x_1, x_2, \cdots)$, in space of infinitely many dimensions, satisfying the inequalities

$$|x_i - a_i| \leqq r_i \qquad (i = 1, 2, \cdots).$$

It is well known that an analogue of the Weierstrass condensation theorem

* Presented under a different title to the American Mathematical Society, December 28, 1916.

† *Transactions of the American Mathematical Society*, Vol. 18 (1917), p. 125. Referred to in the future as Paper 1.

‡ Cf. P. Montel, *Annales de l'École Normale Supérieure*, 3d series, Vol. 24 (1907), p. 265.

§ Loc. cit., p. 264.

holds in R. That is, if $S = (\xi_n;\ n = 1, 2, \cdots)$ is a sequence of points in R, there exists a point $\xi = (z_1, z_2, \cdots)$ and a sub-sequence (ξ'_n) of S such that

$$\lim_{n=\infty} x'_{in} = z_i \qquad [i = 1, 2, \cdots;\ \xi'_n = (x'_{1n}, x'_{2n}, \cdots)].$$

For a sequence of functions $T = [y_n(t);\ n = 1, 2, \cdots]$, the notion of equal continuity has been defined as follows:*

DEFINITION 1. *The functions of T are equally continuous at a point t_0 if, for every $e > 0$ a number $d > 0$ can be assigned so that on the interval $|t - t_0| \leqq d$ the oscillation of every function of T is at most e.*

If T satisfies Definition 1 at all points t_0 on a closed interval (a, b), it is easily established that the functions of T are equally uniformly continuous on (a, b). In the theorem which follows there is stated, without proof, a property of equally continuous functions recognized originally by Arzelà.†

THEOREM I. *Let the functions of T be defined and equally continuous at every point of an interval P which may be finite or infinite, open or closed. If the maxima of the absolute values of the functions $y_n(t)$, for t on P, possess a common finite bound, then, we may extract a sub-sequence from T which converges at every point of P. Moreover, the convergence is uniform on every closed sub-interval of P.*

Consider a sequence $S' = [\xi'_n(t);\ n = 1, 2, \cdots]$ where, for every t on P, $\xi'_n(t)$ is a point in R. Let the sequence formed by the ith coördinates of the functions of S' be represented by $S_i = [x'_{in}(t);\ n = 1, 2, \cdots]$.

THEOREM II. *For every i suppose that the functions of S_i are equally continuous on P. Then, we may select a sub-sequence $S = [\xi_n(t)]$ of S' and a point $\eta(t)$ belonging to R for every t, such that, for all t on P,*

$$(2) \qquad \lim_{n=\infty} x_{in}(t) = y_i(t) \qquad (i = 1, 2, \cdots),$$

where $x_{in}(t)$ and $y_i(t)$ are the i-th coördinates of $\xi_n(t)$ and $\eta(t)$ respectively. Moreover, the convergence in (2) is uniform on every closed sub-interval (a, b) of P.

On applying Theorem I to the sequence S_1, it is seen that we may select a sub-sequence $S^{(1)} = [\xi_n^{(1)}(t)]$ from S' corresponding to which a function $y_1(t)$ exists satisfying

$$(3) \qquad \lim_{n=\infty} x_{1n}^{(1)}(t) = y_1(t),$$

uniformly on (a, b), where $x_{1n}^{(1)}(t)$ is the first coördinate of $\xi_n^{(1)}(t)$. As a result of applying Theorem I to the sequence formed by the second coördinates of the functions of $S^{(1)}$, it follows that a sub-sequence

* Cf. Montel, loc. cit., p. 236.

† Cf. Montel, loc. cit., p. 237.

$S^{(2)} = [\xi_n^{(2)}(t)]$ of $S^{(1)}$ and a function $y_2(t)$ may be selected for which there holds

$$\lim_{n=\infty} x_{in}^{(k)}(t) = y_i(t) \qquad (i = 2,\ k = 2), \tag{4}$$

uniformly for t on (a, b). Obviously, because of (3), condition (4) is also true when $i = 1$. In a similar manner, for every h we obtain a sequence $S^{(h)} = [\xi_n^{(h)}(t)]$ and a function $y_h(t)$ such that the i-th coördinates of $S^{(h)}$ satisfy (4) with $(k = h;\ i = 1, 2, \cdots, h)$. It is easily established that a sequence S satisfying Theorem II is obtained by placing $\xi_n(t) = \xi_n^{(n)}(t)$, and by identifying $y_i(t)$ of (2) with that in (4). In view of (2), it is clear that every $y_i(t)$ satisfies $|y_i(t) - a_i| \leqq r_i$.

3. **The Existence Theorem for System** (1).—In equations (1) suppose that (ξ, t) is in the space U defined by the condition (ξ *in* R, $a \leqq t \leqq b$).

DEFINITION* 2. *A function $f(\xi)$, defined in R, is completely continuous at a point ξ in R if, whenever* $\lim_{n=\infty} x_{in} = x_i$ $(i = 1, 2, \cdots)$, *it follows that*

$$\lim_{n=\infty} f(\xi_n) = f(\xi) \qquad [\xi_n = (x_{1n}, x_{2n}, \cdots), \textit{ in } R].$$

It is well known that a function $f(\xi)$, completely continuous at all points of R, possesses a maximum and a minimum, each of which is attained at least once in R. If, for every t, $\xi(t)$ is a point in R whose coördinates $x_i(t)$ are continuous functions of t, it is easily verified that $f[\xi(t)]$ is a continuous function of t. The author has proved† the simple result that, if $(b_1, b_2, \cdots)$ is a sequence of positive numbers and if $e > 0$ is assigned, a number $d > 0$ can be found such that if (ξ', ξ'') are two points of R whose ith coördinates satisfy $|x_i' - x_i''| \leqq db_i$ $(i = 1, 2, \cdots)$, then

$$|f(\xi') - f(\xi'')| \leqq e. \tag{5}$$

In the future let us assume that f_i in (1) satisfies the following conditions:

H_1. If $[(\xi_n, t_n);\ n = 1, 2, \cdots]$ is a sequence of points in U for which

$$\lim_{n=\infty} t_n = t, \qquad \lim_{n=\infty} x_{in} = x_i \qquad (i = 1, 2, \cdots),$$

it follows that $\lim_{n=\infty} f_i(\xi_n, t_n) = f_i(\xi, t)$.

H_2. There exists a number $M > 0$ such that, for every value of i, the maximum m_i of $|f_i(\xi, t)|$ in the region U satisfies $m_i \leqq r_i M$.

THEOREM III. *Suppose that $a \leqq t_0 < b$. Then, there exists at least one function $\xi(t)$ whose coördinates $x_i(t)$ fulfill the initial conditions ($x_i(t_0) = a_i$; $i = 1, 2, \cdots$) and satisfy (1) for $t_0 \leqq t \leqq c$ where c is algebraically the smaller of b and $(t_0 + 1/M)$.*

In order to prove the theorem let us extend the notion of a Cauchy

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* Cf. Paper I, p. 129.

† Paper I, p. 130.

polygon,* as defined for a finite system of differential equations, to the case of an infinite system (1).

Let p represent a partition of the values $t \geqq t_0$ with the division points $(t_0, t_1, \cdots, t_m = b)$. Then, the coördinates $b_i(t)$ of the Cauchy polygon $\beta(t)$ for (1), corresponding to the partition p and satisfying $\beta(t_0) = \alpha = (a_1, a_2, \cdots)$, is defined by the following equations, where $i = 1, 2, \cdots$:

$$(6)\qquad \begin{cases} b_i(t) = a_i + f_i(\alpha, t_0)(t - t_0) & (t_0 \leqq t \leqq t_1), \\ b_i(t) = b_i(t_1) + f_i(\beta_1, t_1)(t - t_1) & [t_1 \leqq t \leqq t_2;\ \beta_1 = \beta(t_1)], \\ \quad\cdot \qquad\qquad \cdot & \qquad\qquad \cdot \\ \quad\cdot \qquad\qquad \cdot & \qquad\qquad \cdot \\ \quad\cdot \qquad\qquad \cdot & \qquad\qquad \cdot \\ b_i(t) = b_i(t_h) + f_i(\beta_h, t_h)(t - t_h) & [t_h \leqq t \leqq t_{h+1};\ \beta_h = \beta(t_h)], \\ \qquad\qquad \cdot & \qquad\qquad \cdot \\ \qquad\qquad \cdot & \qquad\qquad \cdot \\ \qquad\qquad \cdot & \qquad\qquad \cdot \end{cases}$$

In order to obtain the sequence of polygons which will later be shown to converge to a solution of (1), select a sequence of partitions $(p'_n;\ n = 1, 2, \cdots)$ with norms (d'_n). Suppose that $\lim_{n=\infty} d'_n = 0$. To each partition p'_n there corresponds by (6) a polygon $\beta'_n(t)$. Theorem III will be proved by showing that the sequence $S' = (\beta'_n(t))$ satisfies the hypotheses of Theorem II and by then establishing the fact that the limit function given by that theorem is a solution of (1).

First consider some properties possessed in common by all Cauchy polygons. For every partition p, it follows from (6) and H_2 that, if t is on the interval (t_h, t_{h+1}), where $t_{h+1} \leqq c$, then,

$$(7)\qquad \begin{aligned} |b_i(t) - a_i| &\leqq |b_i(t) - b_i(t_h)| + |b_i(t_h) - b_i(t_{h-1})| + \cdots + |b_i(t_1) - b_i(t_0)| \\ &\leqq r_i M\{ |t - t_h| + |t_h - t_{h-1}| + \cdots + |t_1 - t_0| \} = r_i M(t - t_0) \leqq r_i. \end{aligned}$$

Consequently, for all t on (t_0, c), the polygon $\beta(t)$ is defined and in R. In the future suppose that t is on the interval (t_0, c). In the same manner as we obtained (7) it is verified that, for every partition p, and for all points (t', t'') on interval (t_0, c),

$$(8)\qquad |b_i(t') - b_i(t'')| \leqq Mr_i |t' - t''|.$$

Hence, if $e > 0$ is assigned, a number $w > 0$ can be found so that, for every partition p and for every i,

$$(9)\qquad |b_i(t') - b_i(t'')| \leqq er_i,$$

provided that $|t' - t''| \leqq w$.

Consider the sequence S'. As a consequence of (7) and (9) it follows

* Cf. Bliss, Princeton Colloquium, p. 89.

that S' satisfies the hypotheses of Theorem II on the interval (t_0, c). Let $S = (\beta_n(t))$ be the sub-sequence of S' and let $\xi(t)$ be the point given by Theorem II in the present instance. Equation (2) becomes

$$\lim_{n=\infty} b_{in}(t) = x_i(t) \quad [\beta_n(t) = (b_{1n}(t), b_{2n}(t), \cdots); \; i = 1, 2, \cdots],$$

uniformly for t on (t_0, c). As a consequence of (6) it is obvious that $\xi(t_0) = \alpha$. In view of (9) it is seen that, if $e > 0$ is assigned, a number $w > 0$ may be found such that, if $|t' - t''| \leqq w$, then

$$|x_i(t') - x_i(t'')| \leqq er_i \qquad (i = 1, 2, \cdots). \tag{10}$$

It remains for us to show that $\xi(t)$ is a solution of (1).

As a consequence of (6) we may state that the coördinates $b_{in}(t)$ of $\beta_n(t)$ satisfy

$$b_{in}(t) = a_i + \int_{t_0}^{t} f_i[\beta_n(\bar{t}), \bar{t}]dt,$$

where $\bar{t}$ is a function of the variable t of integration and is defined as the last division point preceding t in the partition p_n corresponding to $\beta_n(t)$. Let us show that, for every value of i,

$$\lim_{n=\infty} f_i[\beta_n(\bar{t}), \bar{t}] = f_i[\xi(t), t], \tag{11}$$

uniformly for t on (t_0, c). When this has been proved, we may write

$$x_i(t) = a_i + \int_{t_0}^{t} f_i[\xi(t), t]dt,$$

from which it follows by differentiation that $\xi(t)$ is a solution of (1).

To establish (11) let us consider

$$\begin{aligned} &|f_i[\beta_n(\bar{t}), \bar{t}] - f_i[\xi(t), t]| \\ &\quad \leqq |f_i[\beta_n(\bar{t}), \bar{t}] - f_i[\beta_n(t), t]| + |f_i[\beta_n(t), t] - f_i[\beta_n(t'), t']| \\ &\qquad + |f_i[\beta_n(t'), t'] - f_i[\xi(t'), t']| + |f_i[\xi(t'), t'] - f_i[\xi(t), t]|, \end{aligned} \tag{12}$$

where t' is any point on (t_0, c). In the function $f_i(x_1, x_2, \cdots; t)$ we may think of $(x_1, x_2, \cdots; t)$ as being the coördinates in the space of infinitely many dimensions defined by $(|x_i - a_i| \leqq r_i; \; t_0 \leqq t \leqq c)$. As a consequence of (5), if $e > 0$ is assigned, a number $w > 0$ can be determined so that, if $(|x_i' - x_i''| \leqq wr_i)$ and $|t' - t''| \leqq w$,

$$|f_i(\xi', t') - f_i(\xi'', t'')| \leqq e. \tag{13}$$

In (12) recall that $|\bar{t} - t| \leqq d_n$, the norm of the partition p_n, and, therefore, as a result of (8),

$$|b_{in}(\bar{t}) - b_{in}(t)| \leqq d_n M r_i.$$

Therefore, it follows from (13) that the first term on the right in (12) approaches zero for $n = \infty$, uniformly for all t on (t_0, c). In view of (9)

and (10), an application of (13) to the second and fourth terms on the right of (12) shows that, if an $e > 0$ is assigned, a number $g > 0$ can be found so that, if $|t - t'| \leqq g$, then each of these terms will be at most $e/3$, for all values of n.

Before considering the third term in (12), let us form a partition ($v_0 = t_0$, $v_1, \cdots, v_m = c$), of the interval (t_0, c), with the number g of the previous paragraph as its norm. Since $f_i(\xi, t)$ is completely continuous and since

$$\lim_{n=\infty} b_{in}(t) = x_i(t),$$

an integer N may be determined so that

$$|f_i[\beta_n(v_j), v_j] - f_i[\xi(v_j), v_j]| \leqq \frac{e}{3} \quad (j = 0, 1, \cdots, m;\ n \geqq N).$$

In (12), for a fixed value of t, suppose that t' is one of the (v_j) differing from t by at most g. Then, if $n \geqq N$, the sum of the last three terms in (12) is at most e. Since N was chosen independently of t, it is seen that the uniformity stated in (11) has been completely established. Hence Theorem III has been proved.

Theorem III was concerned with the existence of a solution on the interval $t \geqq t_0$. A similar thoerem may be stated for the interval $a \leqq t \leqq t_0$.

In paper I the author established* the existence of a unique solution of (1) under the conditions of Theorem III together with an additional assumption. The proof of the uniqueness could be made in the same way in the present paper with the results of Theorem III as a starting point.

UNIVERSITY OF MINNESOTA,
July 1, 1920.

* Loc. cit., p. 144.